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GRAVITY-INDUCED SPREADING OF A DROP OF A VISCOUS FLUID OVER A SURFACE

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The unsteady-state nonlinear problem of spreading of a drop of a viscous fluid on the horizontal surface of a solid under the action of gravity and capillary forces is considered for small Reynolds numbers. The method of asymptotic matching is applied to solve the axisymmetrical problem of spreading when the gravity exerts a significant effect on the dynamics of the drop. The flow structure in the drop is determined at large times in the neighborhood of a self-similar solution. The ranges of applicability of the quasiequilibrium model of drop spreading with a dynamic edge angle and a self-similar solution are found. It is shown that the transition from one flow model to another occurs at very large Bond numbers.

The theory of drop spreading is based on the model of a nonequilibrium (dynamic) edge angle [1, 2] and takes into account the possible precursive film [2–4] moving ahead of the visible edge of the drop. The gravity effect on a viscous flow in the drop was considered by the author [5] in a quadratic approximation relative to the Bond number (B), which holds for B < 20. In the present paper, in contrast to [5], the viscous flow is determined for any Bond numbers. The theoretical models of the dynamics of wetting [1, 2] that we use are supported by the data of many experiments [6–8] in the absence of gravity.

1. Basic Equations. We consider the spreading of a drop of a viscous fluid on the smooth horizontal surface of a solid. For quite large times from the beginning of the spreading, the Reynolds numbers and the capillary number (Ca) are small. Here, the angle of slope of the gas–fluid interface is small ($\alpha \ll 1$). In the region of sufficiently large thicknesses of the layer h (the drop thickness at the center is assumed to be a macroscopic quantity), the hydrodynamic description is valid.

For the case of a horizontal surface, with allowance for capillary forces and gravity, the creeping motion of a thin layer at a sufficient distance from the contact line L_0 (for relatively large thicknesses h of the layer) is described by the equation in coordinates on this surface:

$$\frac{\partial h}{\partial t} = -\frac{1}{3\mu}\,\mathrm{div}\{h^3\,\mathrm{grad}\,(\sigma\Delta h - \rho g h)\}. \tag{1.1}$$

Here ρ is the density, g is the acceleration of gravity, μ and σ are the coefficients of dynamic viscosity and surface tension. In close proximity to the contact line of three phases, the asymptotic behavior of the angle of slope of the interface 1

$$\alpha^3 = 9\mathrm{Ca}\,(s - (1/3)\ln s + (\ln s - 4)/(9s) + \ldots), \quad \alpha = |\nabla h|, \quad \mathrm{Ca} = \mu v/\sigma,$$
$$s = \ln(h/h'_m) + C, \qquad |s| \gg 1, \qquad |\mathrm{Ca}| \ll 1, \tag{1.2}$$

where v is the velocity of the contact line, is observed [1, 2]. The parameter h'_m corresponds to the minimum characteristic thickness at which the asymptotic behavior occurs. The method of determining the constant asymptotic behavior ($C - \ln h'_m$) was indicated in [1, 2, 5]. In connection with the unbounded spreading of a drop considered below, the case of a zero static edge angle of wetting is of most interest. For the case of

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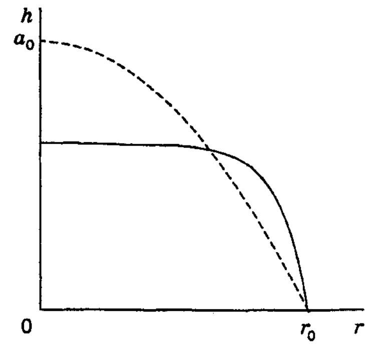

Fig. 1

complete wetting, the constant $C = 0$, and, if there is a precursive wetting film moving under the van der Waals forces, the minimum scale in (1.2) is equal to the maximum thickness of this film [2]:

$$h'_m = 1.085(3\mathrm{Ca})^{-1/3}(A'/(2\pi\sigma))^{1/2}. \tag{1.3}$$

A coefficient equal to 1.085 was found from numerical calculation of the corresponding boundary-value problem [2]. If the capillary numbers or the angles α are insufficiently small (generally speaking, their smallness is not sufficient compared with unity), the value of h'_m formally calculated using (1.3) does not exceed the molecular sizes and there is no precursive film [2, 4]. Here h'_m has the order of the molecular size [1].

For the problem of the dynamics of a film considered at a large distance from the contact line of three phases (the external problem), it is convenient to write the condition at the contact line L_0 [for the points of which the external solution is $h(x_0) = 0$] in asymptotic form [5]:

$$\begin{gathered} \alpha^3 = \alpha^3_{(0)}(h_0) + 9\mathrm{Ca}\Big(1 - \frac{1}{3s_0}\Big)\ln\Big(\frac{h}{h_0}\Big) + \ldots, \qquad x \to x_0, \\ s_0 = \ln\Big(\frac{h_0}{h'_m}\Big) + C, \qquad \ln\Big(\frac{h_0}{h}\Big) \ll s_0, \qquad s_0 \gg 1. \end{gathered} \tag{1.4}$$

The parameter h_0 is the characteristic thickness of the layer for the external problem. One can determine this parameter as follows. We note that the quantity s_0 should not differ greatly from the characteristic (maximum) value of s in the transient region between the internal and external regions, which limits from above the range of s values to which (1.2) and (1.4) are applicable. This condition is used to refine the limit of applicability of the asymptotic solution of the external problem upon spreading of the drop.

2. Dynamics of a Heavy Drop for Large Bond Numbers. In the axisymmetrical case, Eq. (1.1) is written together with the regularity conditions on the axis of symmetry and the condition of decrease in the solution on the contact line:

$$\begin{gathered} \frac{\partial h}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\Big(\frac{h^3}{3\mu} r \frac{\partial}{\partial r}\Big(\rho g h - \frac{\sigma}{r}\frac{\partial}{\partial r} r \frac{\partial h}{\partial r}\Big)\Big), \\ \frac{\partial h}{\partial r} = 0, \qquad \frac{\partial}{\partial r}\Delta h = 0 \quad \text{for} \quad r = 0, \qquad \frac{h}{h_0} \to 0 \quad \text{for} \quad r \to r_0. \end{gathered} \tag{2.1}$$

Here r is the distance from the axis of symmetry and r_0 is the desired coordinate of the external contact line (the visible edge of the drop). Together with the initial conditions, relations (1.4) and (2.1) determine the drop dynamics. Next, we search for the asymptotic behavior of the solution for sufficiently large times from the moment of beginning of drop spreading when only the drop volume V is significant.

The approximate solution of the flow problem of a drop is known for the case where the effect of viscous forces on the shape of the drop is relatively small in its central region; in a principal approximation, it is determined relative to a small Ca by the balance of the gravity and capillary forces from a static equation. The drop which is quasistatic in its central region forms a dynamic edge angle with a rigid surface. Figure 1 shows the static form of the drop for B = 100 (solid curve) and a parabolic profile for B = 0 (dashed curve). In [5], within the framework of the asymptotic model, an external solution that is close to the quasistatic solution was found with allowance for terms of the form CaB and CaB^2 in the expression for the layer thickness h. In contrast to [5], we search for a solution with allowance for terms of the order of Ca for arbitrary Bond numbers B [9] to expand the range of applicability of the solution.

The solution of the external problem for (2.1) is sought in the form

$$h = h_{(0)}(r,t) + h_{(1)}(r,t) + \ldots, \quad h_{(1)} = \mathrm{Ca}\, f_1(r,t), \quad |h_{(1)}| \ll h_{(0)}, \quad \mathrm{Ca} = \frac{\mu v}{\sigma}, \quad v = \frac{dr_0}{dt}. \tag{2.2}$$

We note that $h_{(0)}$ and f_1 depend on the time only parametrically and do not depend on the velocity (on the number Ca), and $h_{(1)}$ takes into account, in a linear approximation, the effect of viscous flow on the shape of the drop. Substitution of (2.2) into (2.1) and neglect of small Ca^2 allow one to find

$$\frac{\partial^2}{\partial \xi^2}\, \xi\, \frac{\partial h_{(0)}}{\partial \xi} - \frac{\mathrm{B}}{4}\frac{\partial h_{(0)}}{\partial \xi} = 0, \qquad \xi = \frac{r^2}{r_0^2}; \tag{2.3}$$

$$\frac{\partial^2}{\partial \xi^2}\, \xi\, \frac{\partial h_{(1)}}{\partial \xi} - \frac{\mathrm{B}}{4}\frac{\partial h_{(1)}}{\partial \xi} = -F(\xi,t) = -\frac{1}{\xi}\,\frac{3}{16}\,\frac{\mu r_0^4}{\sigma h_{(0)}^3}\int_0^{\xi} \frac{\partial' h_{(0)}}{\partial t}\, d\xi. \tag{2.4}$$

Here $\partial'/\partial t$ is the derivative of the function with respect to r and t, and $\mathrm{B} = \rho g r_0^2/\sigma$.

At the drop edge, the thickness is zero:

$$h_{(0)} = 0, \qquad h_{(1)} = 0, \qquad \xi = 1. \tag{2.5}$$

The regularity condition of the solution at the center requires the boundedness of $\partial h_{(0)}/\partial \xi$ and $\partial h_{(1)}/\partial \xi$ for $\xi = 0$.

We assume that the drop volume V is determined by the unperturbed solution $h_{(0)}$:

$$V = \pi r_0^2 \int_0^1 h_{(0)}\, d\xi = \frac{\pi}{2}\, r_0^2 a_0, \qquad \int_0^1 h_{(1)}\, d\xi = 0. \tag{2.6}$$

The solution of Eq. (2.3) with condition (2.5) is expressed in terms of the modified zero-order Bessel functions of the first kind $I_0(x)$ [5]:

$$h_{(0)} = \frac{a_0 \Omega}{2}\,[I_0(\sqrt{\mathrm{B}}) - I_0(\sqrt{\xi \mathrm{B}})], \quad \Omega^{-1} = I_0(\sqrt{\mathrm{B}}) - \int_0^1 I_0(\sqrt{\xi \mathrm{B}})\, d\xi, \quad \mathrm{B} = \rho g r_0^2/\sigma. \tag{2.7}$$

Using (2.6) and (2.7) and the definition of B, we find the right-hand part of (2.4):

$$F = \frac{3}{8}\,\mathrm{Ca}\, \frac{\alpha_0 r_0^2}{\xi a_0 h_{(0)}^3} \int_0^{\xi} \Big(\frac{a_0}{2} - h_{(0)}\Big) d\xi, \qquad \alpha_0 = -\frac{\partial h}{\partial r}\Big|_{r=r_0} = \frac{\Omega}{2}\sqrt{\mathrm{B}}\, I_0'(\sqrt{\mathrm{B}})\,\frac{a_0}{r_0}. \tag{2.8}$$

Here α_0 is the edge angle of the drop which is not perturbed by a viscous flow $[h = h_{(0)}]$ and I_0' is the derivative with respect to the Bessel function.

Integrating (2.4), we find a solution which is regular for $\xi = 0$:

$$-\frac{\partial h_{(1)}}{\partial \xi} = y_{(0)}\Big[D + \int_0^{\xi} \frac{1}{\xi_2^2 y_{(0)}^2(\xi_2)}\, d\xi_2 \int_0^{\xi_2} \xi_1 y_{(0)}(\xi_1) F(\xi_1)\, d\xi_1\Big], \quad y_{(0)} = -\frac{\partial h_{(0)}}{\partial \xi}, \quad D = \mathrm{const}. \tag{2.9}$$

The time t enters into $F(\xi)$ and the right-hand part of (2.9) as a parameter on which Ca, a_0, α_0, $h_{(0)}$, and $y_{(0)}$ depend. For brevity, the dependence on t in explicit form is omitted.

For $h_{(1)}$, condition (2.6) is easily reduced to the form

$$\int_0^1 \xi \frac{\partial h_{(1)}}{\partial \xi}\, d\xi = 0. \tag{2.10}$$

Excluding D from (2.9) by means of (2.10) and after substitution of (2.8) and certain transformations, including integration by parts, we obtain

$$\frac{\partial h_{(1)}}{\partial \xi} = \frac{\partial h_{(0)}}{\partial \xi} \mathrm{Ca} \frac{3\alpha_0 r_0^4}{8a_0^2} \int_0^{\xi} \Big[\frac{1}{a_0} \int_0^{\zeta} y_{(0)}(\xi_1)\xi_1\, d\xi_1\Big] \frac{W_2(\zeta)}{\zeta^2 y_{(0)}^2(\zeta)}\, d\zeta; \tag{2.11}$$

$$W_2(\xi) = \frac{1}{Y^2} \int_0^{\xi} \Big(\frac{1}{2} - Y\Big) d\xi - \int_0^{\xi} \frac{1}{Y^2}\Big(\frac{1}{2} - Y\Big) d\xi, \qquad Y = \frac{h_{(0)}}{a_0}. \tag{2.12}$$

We consider the solution near the edge of the drop. Letting $\xi \to 1$ ($r \to r_0$) and taking into account that $h = (1-\xi)\alpha_0 r_0/2 + \dots$, we transform (2.11) to the form

$$\alpha = -\frac{dh}{d\xi}\frac{2}{r_0}\sqrt{\xi} = \alpha_0\Big[1 - \frac{3\mathrm{Ca}}{\alpha_0^3}\Big(\ln\Big(\frac{a_0}{h}\Big) - C_1\Big)\Big] + \dots,$$

$$C_1 = -\ln k + \int_0^1 \Big[k^2 W_1 W_2 + \frac{1}{1-\xi}\Big] \frac{W_2(\xi)}{\xi^2 y_{(0)}^2}\, d\xi, \qquad k = \frac{r_0\alpha_0}{2a_0} = \frac{1}{4}\sqrt{\mathrm{B}}\, I_0'(\sqrt{\mathrm{B}})\Omega(\mathrm{B}), \tag{2.13}$$

$$W_1 = \frac{2}{\xi}\Big(-\xi Y + \int_0^{\xi} Y\, d\xi\Big)\Big[\frac{I_0'(\sqrt{\mathrm{B}})}{I_0'(\sqrt{\xi\mathrm{B}})}\Big]^2.$$

In going from (2.11) to (2.13), we separate a part of the integral (2.11) which is divergent as $\xi \to 1$. In accordance with (2.7) and (2.8), the angle α_0 in (2.13) is a known function of the Bond number $\alpha_0(\mathrm{B})$. Together with (2.7), formulas (2.12) and (2.13) complete the solution of the external problem. Using the asymptotic condition (1.4), we find

$$\alpha_0 = \alpha_{(0)}(h_0), \qquad h_0 = a_0 \exp(-C_1), \tag{2.14}$$

where $\alpha_{(0)}(h_0)$ is the asymptotic behavior of (1.2) written with accuracy s_0^{-2}. As a result, for a completely wetted surface, it follows that $\alpha_0^3(\mathrm{B}) = 9\mathrm{Ca}(s_0 - (1/3)\ln s_0)$ and $s_0 = \ln(h_0/h_m')$. The corresponding equation of variation in the radius r_0 of the drop foundation is written in the form

$$\frac{\mu}{\sigma}\frac{dr_0}{dt} = \frac{\alpha_0^3(\mathrm{B})}{9\ln(h_0/h_m') - 3\ln\ln(h_0/h_m')}, \quad a_0 = \frac{2V}{\pi r_0^2}, \quad \mathrm{B} = \frac{\rho g r_0^2}{\sigma};$$

the function $\alpha_0(\mathrm{B})$ is defined by (2.8), h_0 is known from (2.14), and $C_1 = C_1(\mathrm{B})$. Expressing r_0 in terms of α_0, one can write an equation for α_0. Integration of this equation faces no difficulties and gives $r_0(t)$ and $\alpha_0(t)$.

When the gravity is insignificant ($\mathrm{B} = 0$) and the drop in the central region is close to a spherical segment, formula (2.13) gives $C_1 = 2$. In [5], corrections of the order of B^2 in (2.13) are found. Formula (4.12) in [5], which is similar to (2.13), has the form

$$\alpha^3 = \alpha_0^3(\mathrm{B}) + 9\mathrm{Ca}\Big[2 + \ln(1-\xi) - \frac{5}{4}\frac{\mathrm{B}}{24} + \frac{7}{6}\Big(\frac{\mathrm{B}}{24}\Big)^2 + \dots\Big].$$

As a result, for the constant (2.13), it follows that

$$C_1 = 2 - \frac{5}{4}\frac{\mathrm{B}}{24} + \frac{7}{6}\Big(\frac{\mathrm{B}}{24}\Big)^2 - \ln k + \dots . \tag{2.15}$$

The calculation results for $C_1(\mathrm{B})$ are shown in Fig. 2 [the solid curve corresponds to (2.13), and the dashed curve to (2.15)]. In the range of moderate values of the Bond number ($\mathrm{B} < 20$), Eqs. (2.13) and (2.15)

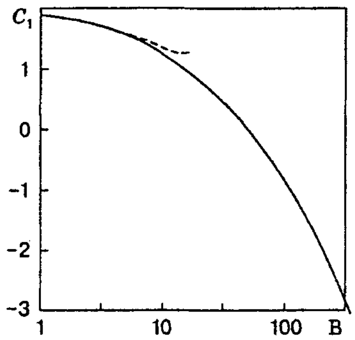

Fig. 2

agree well. For large values (B $\gg$ 20), calculation with the use of (2.13) gives negative values of $C_1 < 0$. The increase in h_0 relative to (2.14) because of the monotone decrease in C_1 reflects a greater role of the viscous-energy dissipation in the external flow region relative to the basic dissipation near the contact line of three phases.

Let us estimate the Bond numbers at which the solution obtained and the quasiequilibrium model hold. Taking into account (2.14) and the remark to (1.3), the condition of applicability of the theory can be written in the form $|C_1| \ll s_0 + C_1 = \ln(a_0/h_m)$. Generally, we have $\ln(a_0/h_m) \sim 10$ in experiments with macroscopic drops [1, 6–8]. If the limit of applicability $|C_1| \sim 0.3\ln(a_0/h_m)$ is taken for estimation, the model can be used up to $C_1 \sim -3$, to which corresponds a critical number B_* of the order of 300. The characteristic critical number depends relatively weakly on the drop size and cannot exceed 500. The range of applicability of the model considered with respect to the number B is appreciably smaller for droplets with a height at the center of the order of 10^{-3} cm.

The possibility to apply the quasiequilibrium flow model of a drop up to Bond numbers of approximately several hundreds does not contradict the estimate [5] of the loss of its applicability in the region B > 24 and also increases its importance for the description of the process of spreading of a heavy drop. To convince oneself that the estimates are correct, it is advisable to consider the limiting case B $\to \infty$ ($t \to \infty$), where the capillary forces give a minor contribution to drop spreading compared with the gravity.

3. Gravity Force-Induced Spreading of a Drop Close to Self-Similar Spreading. For a sufficiently large radius of the drop r_0, the effect of capillary forces on the flow in its central region should be negligible. It is of interest to find the influence of capillary forces on the flow structure as B $\to \infty$ and to determine a bound for the number B above which the corresponding limiting solution can be valid.

In the absence of capillary forces ($\sigma = 0$), Eq. (2.1) has a self-similar solution of the form [10]

$$h = \frac{1}{t_1^{1/4}} f\left(\frac{r}{t_1^{1/8}}\right), \qquad t_1 = \frac{\rho g}{3\mu} t. \tag{3.1}$$

The solution $h(r,t)$ is localized in the finite region $r < r_0$:

$$h = (f_0/t_1^{1/4})(1 - r^2/r_0^2)^{1/3}, \quad r_0 = (4/\sqrt{3})\, f_0^{3/2} t_1^{1/8}, \quad (3\pi/4)\, h(0,t_1) r_0^2 = V, \quad 4\pi f_0^4 = V. \tag{3.2}$$

We shall seek the solution of Eq. (2.1) as $t \to \infty$, which is close to the exact solution (3.1), (3.2) in the region with the excluded small neighborhood of the drop edge ($r = r_0$). In the small neighborhood of the edge, the solution is quasistationary and depends on the variable $r - r_0(t)$. The solution (3.2) can be written in the form

$$h = \left(\frac{9\mu v}{\rho g}\right)^{1/3} \left(\frac{r_0^2 - r^2}{2r_0}\right)^{1/3}, \qquad v = \frac{dr_0}{dt}.$$

As a result, the quasistationary dependence

$$r \to r_0, \qquad h = \left(\frac{9\mu v}{\rho g}\right)^{1/3} (r_0 - r)^{1/3}, \qquad v = \frac{dr_0}{dt} \tag{3.3}$$

follows for the small neighborhood of the drop edge.

We consider (3.3) as a condition at the boundary of the small neighborhood $r = r_0$, inside which the dynamics of the film is described by the complete equation (2.1) ($\sigma \neq 0$). In this neighborhood, the solution of the form $h(r - r_0, t)$ satisfies the quasistationary equation

$$-v \frac{\partial h}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r}\left(\frac{h^3}{3\mu} r \frac{\partial}{\partial r}\left(\rho g h - \frac{\sigma}{r}\frac{\partial}{\partial r} r \frac{\partial h}{\partial r}\right)\right). \tag{3.4}$$

Leaving in (3.4) the principal terms, which correspond to a plane-problem approximation justified for $r \sim r_0$, for $r_0 - r \ll r$, we obtain

$$\frac{\partial^3 h}{\partial r^3} - \frac{\rho g}{\sigma}\frac{\partial h}{\partial r} = \frac{3\mathrm{Ca}}{h^2}. \tag{3.5}$$

Let us introduce the scales along the h and r axes,

$$H = L(3\mathrm{Ca})^{1/3}, \qquad L = (\sigma/(\rho g))^{1/2}, \tag{3.6}$$

and rewrite (3.5) in dimensionless form

$$\frac{\partial^3 y}{\partial \zeta^3} - \frac{\partial y}{\partial \zeta} = \frac{1}{y^2} \qquad \left(y = \frac{h}{H}, \qquad \zeta = \frac{r - r_0}{L}\right). \tag{3.7}$$

According to (3.3), the solutions of Eq. (3.7) should satisfy the condition

$$y = 3^{1/3}(-\zeta)^{1/3} + \ldots \quad \text{as} \quad \zeta \to -\infty, \tag{3.8}$$

which is justified for large B.

For relatively small values of y, the term $-\partial y/\partial \zeta$ in (3.7) is not significant, and the asymptotic solution (1.2), from which it follows in the principal approximation that

$$\frac{\partial y}{\partial \zeta} = -(3)^{1/3}(\ln(y/y_m))^{1/3}, \quad y_m = h'_m/H, \quad y \ll 1, \quad \ln(y/y_m) \gg 1, \tag{3.9}$$

is the approximate solution.

We denote the characteristic value of y at which the transition from solution (3.8) to solution (3.9) occurs by y_+. Taking into account that $\ln(y_+/y_m) \gg 1$ (this corresponds to the condition of validity of (3.9)], the passage from (3.9) to (3.8) can be approximately described, ignoring the right-hand side of (3.7), by means of the equilibrium equations with the boundary conditions

$$\bar{y}''' - \bar{y}' = 0; \quad \bar{y} = 0, \quad \zeta = \zeta_+; \qquad \bar{y} \to y_+, \quad \zeta \to -\infty, \tag{3.10}$$

where ζ_+ is the desired quantity. The solution (3.10)

$$\bar{y} = y_+(1 - \exp(\zeta - \zeta_+)) \qquad (\zeta < \zeta_+) \tag{3.11}$$

describes a semi-infinite film of constant thickness y_+ far from $\zeta = \zeta_+$. Formula (3.11) is correct under the restriction $y_+^3 \exp(\zeta - \zeta_+) \gg (1 - \exp(\zeta - \zeta_+))^{-2}$, under which $|\bar{y}'| \gg y^{-2}$ and (3.10) holds. If $y_+ \gg 1$, for $\zeta - \zeta_+ \sim 0$, the restriction $y^2 \gg 1/y_+$ is sufficient for (3.11) to be the approximate solution of (3.7). Assuming that $y_+ \gg 1$, we calculate $\bar{y}' = -y_+$ for $\zeta \to \zeta_+$ from (3.11). Equating this quantity to the approximately constant value of (3.9) (matching the angles of slope), we obtain

$$y_+ = 3^{1/3}(\ln(y_+/y_m))^{1/3} \gg 1 \qquad (y_m = h'_m/H). \tag{3.12}$$

This equation is easily solved by iteration, because $\ln(1/y_m) \gg 1$.

To match (3.11) and (3.8), we consider (3.11) as a solution in the boundary layer for Eq. (3.7). Indeed, if $y \to \infty$ in (3.8), the scale ζ, on which y varies in (3.8), also increases infinitely ($dy/d\zeta \to 0$ and $y \to \infty$).

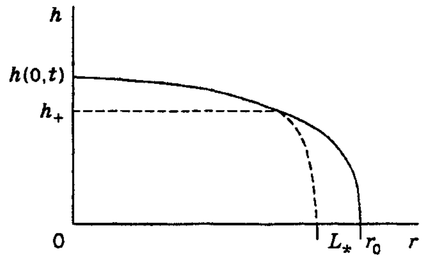

Fig. 3

Therefore, for $y = y_+ \gg 1$, one can consider (3.8) as an external solution [of a greater scale compared with the scale of (3.11)], and (3.11) to be a boundary-layer solution. From (3.8) and (3.12), we find

$$|\zeta_+| = (1/3)y_+^3 = \ln(y_+/y_m). \tag{3.13}$$

According to (3.6)–(3.8) and (3.13), the self-similar solutions (3.1) and (3.2) are not applicable to the following small neighborhood of the line of wetting:

$$r_0 - r < L_*, \qquad L_* = L|\zeta_+| = (\sigma/(\rho g))^{1/2}\ln(y_+/y_m); \tag{3.14}$$

note that, for $r = r_0 - L_*$, the self-similar profile of the thickness (3.2) drops abruptly to zero for the critical thickness (Fig. 3):

$$h_+ = (\sigma/(\rho g))^{1/2}(9\text{Ca})^{1/3}\ln(h_+/h'_m)^{1/3} = \alpha_0(\sigma/(\rho g))^{1/2}. \tag{3.15}$$

In Fig. 3, the drop profile is approximated by the solid curve for $h > h_+$, which corresponds to (3.2), and by the dashed curve for $h < h_+$. The value of r_0 corresponds approximately to the second formula in (3.2).

The quantity α_0 is the dynamic edge angle of the quasiequilibrium segment of the film (3.11) on which the profile (3.2) cuts off. This segment of the film profile is located between two regions in which the effect of film dynamics is important. The resulting formula describes the effect of the truncation of the self-similar profile at the drop edge under the action of capillary forces. The distance of the truncation L_* depends weakly on the time. The truncation thickness referred to the thickness at the center decreases as the drop radius in the power $-1/3$.

The effect of the capillary truncation of the self-similar profile (3.2) allows one to estimate conditions under which this solution is reached as $t \to \infty$. Obviously, the solution holds only when the length L_* of the truncated segment of the profile on (3.14) is small in comparison with the droplet radius:

$$r_0 \gg L_* \quad \text{if} \quad \text{B}^{1/2} \gg \ln(h_+/h'_m). \tag{3.16}$$

We note that, for large enough drops, for which the influence of gravity and the conditions of terrestrial experiments are important, the quantity $\ln(h_+/h'_m)$ is of the order 8–10.

With allowance for the large value of $\ln(h_+/h'_m)$, it follows from (3.16) that the capillary forces affect strongly the flow structure in a drop spreading primarily under the action of the gravity. Owing to the effect of the capillary truncation of the gravitationally viscous profile of the drop in the neighborhood of the line of wetting ($h = 0$), solution (3.2) is valid only for very large values of the Bond number, $\text{B} \gg 100$ (for $\text{B} > 1000$ in reality). It is in agreement with the results of Sec. 2 and is important for understanding the conclusion on the applicability of the quasiequilibrium (or updated) model of drop spreading up to Bond numbers of the order of several hundreds. For droplets (of thickness smaller than 0.001 cm at the center), the values of $\ln(h_+/h'_m)$ can be noticeably smaller than 10, and, hence, the model of a flow determined by the dynamic edge angle holds for smaller values of B.

Just as near water, for the case ρ and σ, the condition $\text{B} \gg 100$ is equivalent to $2r_0 \gg 6$ cm. The diameter of the foundation of a drop should be tens of centimeters for the flow in it to be described by the self-similar solution (3.2).

If the radius of the drop foundation is approximately 1 cm in the experiment, solution (3.2) is not applicable to its description. In [10], the self-similar solution (3.2) was used for interpreting experiments with relatively small drops ($r_0 < 1$ cm). It follows that all the experiments considered in [10] are concerned with the case where the model of a quasistatic (in the central region) drop with a dynamic edge angle is valid [1, 2].

The flow in a quite large drop consists of a large number of characteristic regions, in which various forces show up. The asymptotic description of these regions in a thin drop (film) is possible due to the large parameter s_0 (1.4). The value of (h'_m) in (3.9) and (3.15) can be determined by van der Waals forces by means of (1.3) (the maximum thickness of a precursive film). Here, for $h > h'_m$, the capillary forces are important; in the region of small thicknesses $h \ll h'_m$, the film moves under the action of van der Waals forces, and the flow can be significantly nonstationary (the appropriate mathematical model is given in [11]).

Conclusions. (1) The solution of the problem of drop spreading has been found in the second approximation relative to a small capillary number Ca for arbitrary Bond numbers. The solution describes a spreading regime which is determined by the dynamic edge angle when the viscous flow plays a basic role near the drop edge. The solution is valid for $B < 300$ (up to drop diameters of the order of 10 cm).

(2) The flow structure is determined on large times, when the solution is close to a self-similar solution. The capillary forces strongly affect the flow and shape of a macroscopic drop up to Bond numbers of the order of 10^3 or greater. Capillary truncation of the self-similar profile of the drop at its edge occurs. It is important that the estimates of the radii of the drop (tens of centimeters) at which the solution on large times becomes closer to a self-similar solution agree well with the estimates of the radii at which the quasiequilibrium model of a drop with a dynamic edge angle works.

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